Solution to the Problem of the Risk-Sensitive Optimal Estimator over Nonlinear Observations

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Abstract— **The optimal estimator problem is considered for stochastic systems with polynomial linear drift terms with polynomial observations and intensity parameters multiplying diffusion terms in the state and observation equations. The estimator equations are obtained using a value function as solution to the corresponding nonlinear parabolic equation PDE. The performance of the obtained risk-sensitive estimator stochastic systems with polynomial linear drift terms with polynomial observations is verified in a numerical example, through comparing the mean-square criteria values for the optimal risk-sensitive estimator and polynomial filtering equations. The simulation results reveal strong advantages in favor of the designed risk-sensitive equations in regard to the final criteria values for some values of the parameter ϵ.**

Keywords: error, criterion, filter desing, filtering problems, estimators.

I. INTRODUCTION

In (Basin, 2003), (Basin and Martínez, 2004), (Basin, Perez and Calderón, 2008) and (Basin and Darío, 2009), has continued the investigation of optimal filtering for polynomial systems. The cubic sensor problem ((Hazewinkel, Marcus and Sussmann, 1983)) is one of the applications you have in solving the problem of optimal filtering for polynomial observations, where there is a great advantage in theory and practice of filtering.

More than thirty years ago, Mortensen (Mortensen, 1968) introduced a deterministic filter model which provides an alternative to stochastic filtering theory. In this model, errors in the state dynamics and the observations are modeled as deterministic "disturbance functions", and an exponential mean-square cost criterion disturbance error is to be minimized. Special conditions are given for the existence, continuity and boundedness of $f(x(t))$ in the state equation, which is considered nonlinear, and the linear function $h(x(t))$ in the observation equation. A concept of the stochastic risk-sensitive estimator, introduced more recently by McEneaney (McEneaney, 1998), regard a dynamic system where $f(x)$ is a nonlinear function and linear observations and existence of parameter $\tilde{\epsilon}$ multiplying diffusion term in both equations (state and observations). In (M. V. Basin and Darío Calderón Álvarez, 2009) obtained the equations of optimal filtration for polynomial systems over polynomial observations. The goal of this work is to obtain the optimal filter risk-sensitive equations when the form of $f(x(t))$ is polynomial of first degree and the observation equations are polynomial of second degree, the parameter $\tilde{\epsilon}$ multiplies the diffusion term in the state and observations equations. This filtering equations are obtained taking a value function as solution of the parabolic partial differential equation and mean-square exponential criterion to be minimized. These equations are compared with Polynomial filter equations in an example, for certain values of the parameter ϵ, and the same exponential mean-square cost criterion values. It was proved, the performance of the risk-sensitive Optimal Estimator equations versus Polynomial filtering equations for systems of first degree with observation of second degree. This performance is shown verified in a numerical example against the mean-square optimal for Polynomial filter, through comparing the exponential mean-square criterion values. The simulation results reveal strong advantages in favor of the designed risk-sensitive equations for some values of the intensity parameters multiplying diffusion terms in state and observation equations. Tables of the criteria values and simulation graphs are included. This work is organized as follows: The filtering problem statement is presented in Section II. In Section III is presented the solution to optimal risk-sensitive estimator problem for polynomial systems over polynomial observations. An numerical example is solved for the risk-sensitive optimal filter algorithms and Polynomial Filter in Section IV. In Section V are the conclusions.

II. FILTERING PROBLEM STATEMENT

Consider the following stochastic model (1), where $X(t)$ denotes the state process. $Y(t)$ denotes a continuous accumulated observations process. $X(t)$ satisfies the diffusion model given by:

$$dX(t) \quad = \quad f(X(t))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}dW(t) \qquad (1)$$

where $f(x(t))$ represents the nominal dynamics, and W is a Brownian motion, and the observation process $Y(t)$ satisfies the equation:

$$dY(t) \quad = \quad h(X(t))dt + \sqrt{\frac{\epsilon}{2\gamma^2}}d\tilde{W}(t), \quad Y_0 = 0, \quad (2)$$

where $h(X(t))$ is a polynomial vectorial function, ϵ is a parameter and W and $\tilde{W}$ are independent Brownian motions, which are also independent of the initial state X_0.

X_0 has probability density $k_\epsilon exp(-\epsilon^{-1}\phi(x_0))$ for some constant k_ϵ.

Let us consider

$$J = \epsilon logE \exp \frac{1}{\epsilon}\int_0^T L(x(t), m(t), t)dt \qquad (3)$$

the quadratic cost criterion to be minimize.

In the rest of the paper the assumptions (A1)-(A4) (from (Fleming and McEneaney, 2001)) are hold:

- (A1)$f, g, h \in \mathbf{R}^n$ with f_x, h_x bounded.
- (A2) $D_1(/x/^2 - 1) \leq \phi(x) \leq D_2(/x/^2 + 1)$.

Here f_x is the matrix of partial derivatives of f with h_x defined similarly. ϕ is a continuous, real-valued function satisfying (A2) for some positive $D_1,\ D_2$.

- (A3) $f,\ h \in \mathbf{R}^n$ with $f,\ h$, bounded and $f_{xx},\ h_{xx}$ bounded and globally Hölder continuous. (A function u is globally Hölder continuous if there exists $\alpha \in (0, 1], K < \infty$ such that $|u(x) - u(y)| \leq K|x - y|^\alpha$ for all $x,\ y$.)
- (A4) Given $R < \infty$, there exists $K_R < \infty$ such that $|\phi(x) - \phi(y)| \leq K_R|x - y|$ for all $|x|, |y| < \infty$.

Let $q(T, x)$ denote the unnormalized conditional density of $X(T)$, given accumulated observations $Y(t)$ for $0 \leq t \leq T$. It satisfies the Zakai stochastic PDE, in a sense made precise, for instance in (Lukes, 1969), sec. 7. Since the normalizing constant k_ϵ above is unimportant for q, it is assumed that

$$\begin{aligned} q(0, x) &= exp(-\epsilon^{-1}\phi(x)) \qquad (4)\\ q(T, x) &= p(T, x)exp[\epsilon^{-1}Y(T) \cdot h(x)] \end{aligned}$$

where $p(T, x)$ is called pathwise unnormalized filter density. Then p satisfies the following linear second-order parabolic PDE with coefficients depending on $Y(T)$.

$$\frac{\partial p}{\partial T} = (\check{L}(T))^* p + \frac{K}{\epsilon}p,$$

Where, for every $g \in \mathbf{R}^n$, let

$$\begin{aligned} Lg &= \frac{\epsilon}{2}tr(ag_{xx}) + f \cdot g_x, \qquad (5)\\ \check{L}(T)g &= Lg - a(Y(T) \cdot h)_x \cdot g_x,\\ K(T, x) &= \frac{1}{2}a(x)(Y(T) \cdot h)_x \cdot (Y(T) \cdot h)_x\\ &\quad -L(Y(T) \cdot h) - \frac{1}{2}|h|^2. \end{aligned}$$

L denote the differential generator of the Markov diffusion $X(t)$ in (1). By assumptions (A1) and (A3) in (W. H. Fleming and W. M. McEneaney, 2001), K is bounded and continuous. $(\check{L}(T))^*$ is the formal adjoint of $\check{L}(T)$. Since $Y_0 = 0, p(0, x) = q(0, x)$. The initial condition for (5) is (4). For some given $Y. \in C_0(0, T]$ (where C_0 denote the space of continuous Y such that $Y_0 = 0$, with the sup norm $\|\ \ \|$). The pathwise filter density p is the unique "strong"solution to (5) and (4) in a sense made precise in (D. L. Lukes, 1969), Sec. 7. Moreover, if we denote dependence on Y. by writing $p(T, x; Y)$, then $p(T, x; \cdot)$, is continuous in the sup norm. Further, p is a classical solution to (5) and (4), with p continuous on $[0, T_1] \times \mathbf{R}^n$ and partial derivatives $p_T,\ \ p_{x_i},\ \ p_{x_i x_j},\ \ i, j = 1, ..., n$ continuous for $0 < T \leq T_1$ ((Yoshida and Loparo, 1989) Chap. 1, and (Ladyženskaja , Solonnikov, Ural'ceva, 1968) Chap. 4). Moreover, $p(T, x; Y.) > 0$. We rewrite (5) as follows:

$$\frac{\partial p}{\partial T} = \frac{1}{2}tr(a(x)p_{xx}) + A \cdot p_x + \frac{B}{\epsilon}p, \qquad (6)$$

where

$$\begin{aligned} A &= -f(x) + a(x)(Y(T) \cdot h(x))_x + \qquad (7)\\ &\quad \epsilon diva_x(x)\\ B(T, x) &= \frac{\epsilon^2}{2}tra_{xx}(x) - \epsilon div[f(x) - a(x)(Y(T) \cdot\\ &\quad h(x))_x] + K(T, x)\\ (diva_x)_j &= \Sigma_{i,j=1}^n (a_{ij})_{x_i},\quad j = 1, ..., n,\\ tra_{xx} &= \Sigma_{i,j=1}^n (a_{ij})_{x_i x_j}. \end{aligned}$$

These assumptions imply uniform bounds for A and B, depending on the sup norm $\|Y.\|$ on $[0, T_1]$, but not on ϵ. Taking log transform: $Z(T, x) = \epsilon logp(T, x)$, which satisfies the nonlinear parabolic PDE

$$\frac{\partial Z}{\partial T} = \frac{\epsilon}{2}tr(Z_{xx}) + A \cdot Z_x + \frac{1}{2}Z_x \cdot Z_x + B, \qquad (8)$$

with initial condition $Z_x(0, x) = -\phi(x)$. The risk-sensitive optimal filter problem consists in found the estimate $C(T)$, of the state $x(t)$ through verification that

$$\begin{aligned} Z(T, x) &= \frac{1}{2}(x - C(T))^T Q(T)(x - C(T)) + \qquad (9)\\ &\quad \rho(T) - Y(T) \cdot h(x(t)) \end{aligned}$$

is a viscosity solution of (12). The notation for all the variables is $x(t) = x(t),\ \ x(t) \in R^n,\ \ w(t) \in R^m,\ \ y(t),\ \ v(t) \in R^p,\ \ f,\ \ h \in R^n$ with $f_x,\ \ h_x$ bounded is assumed throughout. Here h_x is the matrix of partial derivatives of h and the same form for Z_x.

III. Risk-Sensitive Optimal Estimator

Taking $f(X(t)) = A(t) + A_1(t)X(t),\ \ h(X(t)) = E(t) + E_1(t)X(t) + ... + E_n(t)X(t)X^T(t)X(t)X^T(t)...X(t)$, with $A(t) \in R^n,\ \ A_1(t), \in M_{n\times n}, E(t) \in R^p, E_1(t) \in M_{n\times p}, ..., E_n(t) \in T_{n\times n\times n\times n...(nveces)\times n}$ where $M_{i\times j}$ denotes the field of matrices of dimension $i \times j$ and $T_{n\times n...(nveces)\times n}$ denotes the field of tensors of dimension $i \times j \times k$. The following stochastic equations system:

$$\begin{aligned} &dX(t) = A(t) + A_1(t)X(t) + \sqrt{\tilde{\epsilon}}dB(t),\\ &dY(t) = E(t) + E_1(t)X(t) + ... + E_n(t)\cdot\\ &\quad X(t)_{nveces}...X(t) + \sqrt{\tilde{\epsilon}}d\tilde{B}(t), \end{aligned}$$

where $\tilde{\epsilon} = \frac{\epsilon}{2\gamma^2}$.

The filtering problem is to obtain the optimal estimator of the state $X(t)$ given the observations, which minimizes the function exponential cost criterion mean square.

Theorem The solution to the filtering problem, for the system (10) with criterion

$$J = \epsilon log E\{\exp \frac{1}{\epsilon}\int_0^T (X(t)-m(t))(X(t)- m(t))^T/Y(t)\} \quad (10)$$

takes the form:

$$\begin{aligned}\dot{C}(t) &= -Q^{-1}[\dot{Q}(t)C(t)+\dot{Y}(t)E_1(t)-A(t)Q(t) \\ &+A_1(t)Q(t)C(t)+\frac{1}{2}C(t)Q(t)-Q(t)Y(T)\cdot \\ &E_1(t)-2C(t)Q(t)Y(T)E_2(t)-\frac{1}{2}E_1(t)] \quad (11)\\ \dot{Q}(t) &= 2[(\dot{Y}(t))(E_2(t)+E_3(t)X(t)+...+E_n(t)\cdot \\ &X^{n-2}(t))-A_1(t)Q(t)-(Y^T(T)E_1(t)Y(T) \\ &-C(t)Q(t)Y(T))(3E_3(t)+4E_4(t)X(t) \\ &+...+nE_n(t)X^{n-3}(t))-(Y^T(t)E_2(t)\cdot \\ &Y(T)+Y(T)E_2^T(t)Y^T(t)+Q(t)Y(T))(2 \\ &E_2(t)+3E_3(t)+...+nE_n(t)X^{n-2}(t))- \\ &(3Y^T(t)E_3(t)Y(T)+...+nY^T(t)E_n(t)\cdot \\ &Y(T)X^{n-3}(t))(E_1(t)+2E_2(t)+...+ \\ &nE_n(t)X^{n-1}(t))+\frac{1}{2}Q(t)-\frac{1}{2}(\sum_{i=1}^{n}(E_i^2(t)\cdot \\ &X^{2i}(t))+2\sum_{i=0}^{n}E_i(t)(\sum_{j=2}^{n}(E_j(t)X(t)^j)))]\end{aligned}$$

Where $C(t)$ is the vector of estimation of the state with the initial conditions $C(0)=C_0$ and $Q(t)$ is a negative definite symmetric matrix, where the initial condition $Q(0)=q_0$ is derived of the initial conditions for Z. If $\phi(X(t)) = X(t)^T K X(t), \quad Q(0)=-K$.

Proof: The value function is proposed:

$$Z(t,X(t)) = \tfrac{1}{2}(X(t)-C(t))^T Q(t)(X(t)-C(t))+ \rho(T)-Y(T)\cdot h(X(t)),$$

$Z_X(0,X(t)) = -\phi(X(t))$, ($C(t)$, $Q(t)$, $\rho(t)$ are functions defined in $[0,T]$, $C(t)\in \mathbf{R}^n$, $Q(t)$ is a symmetric matrix of dimension $n\times n$ and $\rho(t)$ is a scalar function) as a viscous solution of the equation nonlinear parabolic PDE:

$$\frac{\partial Z}{\partial T} = \frac{\epsilon}{2}tr(Z_{xx})+A\cdot Z_x+\frac{1}{2}Z_x\cdot Z_x+B, \quad (12)$$

Z_X, Z_{XX} are the partial derivatives of Z respect to $X(t)$, and ∇Z is the gradient of Z. The partial derivatives of Z are given by:

$$\begin{aligned}Z_T &= \frac{1}{2}(X(t)-C(t))^T\dot{Q}(X(t)-C(t))- \quad (13)\\ &(X(t)-C(t))^T Q\dot{C}+\dot{\rho}(t)-\dot{Y}h(X(t), \\ Z_X &= \frac{1}{2}Q(t)(X(t)-C(t))+\frac{1}{2}(X(t) \\ &-C(t))^T Q(t)-Y(T)(E_1(t)+2E_2(t)X(t) \\ &+...+nE_n(t)X^{n-1}(t)) \\ Z_{XX} &= Q(t)-Y(T)(2E_2(t)+6E_3(t)X+ \\ &...+n(n-1)E_n(t)X^{n-2}(t)).\end{aligned}$$

Let consider:

$$\begin{aligned}&A = -A(t)-A_1(t)X(t)+Y(T)(E_1(t)+2E_2(t)X(t) \\ &\quad +...+nE_n(t)X^{n-1}(t)), \\ &B = -\epsilon A_1(t)+\tfrac{1}{2}(Y(T)(E_1(t)+2E_2(t)X(t)+... \\ &\quad +nE_n(t)X^{n-1}(t)))^2-(A(t)+A_1(t)X(t))\cdot \\ &\quad [Y(t)((E_1(t)+2E_2(t)X(t)+...+\cdot \\ &\quad nE_n(t)X^{n-1}(t)))]-\tfrac{1}{2}(E(t)+E_1(t)X(t)+... \\ &\quad +E_n(t)X(t)X^T(t)X(t)X^T(t)...X(t))\end{aligned}$$

Substituting (13) and the expressions for A, B in (12), we obtain:

$$\begin{aligned}0 = &-\tfrac{1}{2}(X(t)-C(t))^T\dot{Q}(X(t)-C(t))+(X(t)- \\ &C(t))^T Q\dot{C}-\dot{\rho}+\dot{Y}(E(t)+E_1(t)X(t)+... \\ &+E_n(t)X(t)X^T(t)X(t)X^T(t)...X(t))+\tfrac{\epsilon}{2}tr(Q(t)) \\ &+(-A(t)-A_1(t)X(t)+Y(T)(E_1(t)+2E_2(t)\cdot \\ &X(t)+...+nE_n(t)X^{n-1}(t)))(\tfrac{1}{2}Q(t)(X(t)-C(t)) \\ &+\tfrac{1}{2}(X(t)-C(t))^T Q(t)-(-A(t)-A_1(t)X(t)+ \\ &Y(T)(E_1(t)+2E_2(t)X(t)+...+nE_n(t)X^{n-1}(t))) \\ &+\tfrac{1}{2}(\tfrac{1}{2}Q(t)(X(t)-C(t))+\tfrac{1}{2}(X(t)-C(t))^T Q(t)- \\ &Y(T)(E_1(t)+2E_2(t)X(t)+...+n\cdot E_n(t)X^{n-1}(t))\cdot \\ &(\tfrac{1}{2}Q(t)(X(t)-C(t))+\tfrac{1}{2}(X(t)-C(t))^T Q(t)- \\ &Y(T)(E_1(t)+2E_2(t)X(t)+...+nE_n(t)X^{n-1}(t)))- \\ &\epsilon A_1(t)+\tfrac{1}{2}(Y(T)(E_1(t)+2E_2(t)X(t)+...+nE_n(t) \\ &X^{n-1}(t)))^2-(A_0(t)+A_1(t)X(t))[Y(T)\cdot \\ &(E_1(t)+2E_2(t)X(t)+...+nE_n(t)X^{n-1}(t))] \\ &-\tfrac{1}{2}|E(t)+E_1(t)X(t)+...+E_n(t)X(t)...X(t)|^2.\end{aligned}$$

Collecting the terms containing the factor $X^T(t)X(t)$, $X^T(t)X(t)X^T(t)$ and $X^T(t)X(t)...$ and substituting $X(t)$ for $C(t)$, we obtain the matrix equation for $\dot{Q}(t)$. Collecting the terms containing factor $X(t)$, the vector equations for $\dot{C}(t)$ are obtained (11). $\diamond$

IV. Application Plane in Flight Horizontal

IV-A. Risk-sensitive Optimal Estimator for Polynomial Systems over Polynomial Observations

Consider the differential equations describing the trajectory of a plane that flies describing a circle of radius R at a certain height above sea level in a two-dimensional plane parallel to the plane tangent to the Earth. The plan is to coordinate functions x_1 and x_2, which describe the position of the plane at all times. The control parameter is the function u, which represents the direction of the plane relative to the fixed coordinates (x_1, x_2), which can be changed at will. ((Sira, Márquez, Rivas, Llanes-Santiago, 2005))

The model system is given by:

$$\dot{X}_1(t) = V\cos u + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_1(t), \quad (14)$$
$$\dot{X}_2(t) = V\sin u + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_2(t),$$
$$\dot{Y}(t) = \sqrt{X_1^2 + X_2^2 - R^2} + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_3(t),$$

The output of the system represents the distance to an imaginary circle, plotted on the horizontal plane, centered at the origin and radius R.

Apply the Taylor series to give the form of polynomial equations of state:

$$\dot{X}_1(t) = V + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_1(t), \quad (15)$$
$$\dot{X}_2(t) = Vu + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_2(t).$$

Considering $Z = (\dot{Y} - \sqrt{\frac{\epsilon}{2\gamma^2}}dW_3(t))^2 + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_3(t)$, as the equation of the observation, we have:

$$\dot{Z}(t) = X_1^2 + X_2^2 - R^2 + \sqrt{\frac{\epsilon}{2\gamma^2}}dW_3(t). \quad (16)$$

We can see that the observation equation is of second degree.

Applying the equations (15) and (16) to the system (11), we obtain the equations for risk-sensitive optimal estimator for polynomial observations:

$$\begin{aligned}
\dot{C}_1(t) = {} & (Q_{11}Q_{22} - Q_{12}Q_{21})^{-1}[Q_{22}(\dot{Q}_{11}C_1 + \dot{Q}_{12}C_2 \\
& -Q_{11}V + \tfrac{1}{2}Q_{11}C_1 + \tfrac{1}{2}Q_{12}C_2 - 2Y_T\cdot \\
& Q_{11}C_1 - 2Y_TQ_{12}C_2) - Q_{12}(\dot{Q}_{21}C_1 + \dot{Q}_{22}C_2 \\
& -Q_{21}V + \tfrac{1}{2}Q_{21}C_1 + \tfrac{1}{2}Q_{22}C_2 - 2Y_T\cdot \\
& Q_{21}C_1 - 2Y_TQ_{22}C_2)], \\
\dot{C}_2(t) = {} & (Q_{11}Q_{22} - Q_{12}Q_{21})^{-1}[-Q_{21}(\dot{Q}_{11}C_1 + \dot{Q}_{12}C_2 \\
& -Q_{11}V + \tfrac{1}{2}Q_{11}C_1 + \tfrac{1}{2}Q_{12}C_2 - 2Y_T\cdot \\
& Q_{11}C_1 - 2Y_TQ_{12}C_2) + Q_{11}(\dot{Q}_{21}C_1 + \dot{Q}_{22}C_2 \\
& -Q_{21}V + \tfrac{1}{2}Q_{21}C_1 + \tfrac{1}{2}Q_{22}C_2 - 2Y_T\cdot \\
& Q_{21}C_1 - 2Y_TQ_{22}C_2)], \\
\dot{Q}_{11} = {} & 2\dot{Y} - 8Y_T^2 - 8Q_{11}Y_T - \tfrac{1}{2}(R^4 + C_1^4 + C_1^2C_2^2 - \\
& 2R^2C_1^2 + 2C_1^2), \\
\dot{Q}_{12} = {} & -8Q_{12}Y_T - \tfrac{1}{2}(C_1^3C_2 + C_1C_2^3), \\
\dot{Q}_{21} = {} & -8Q_{21}Y_T - \tfrac{1}{2}(C_1^3C_2 + C_1C_2^3), \\
\dot{Q}_{22} = {} & 2\dot{Y} - 8Y_T^2 - 8Q_{22}Y_T - \tfrac{1}{2}(R^4 + C_2^4 + C_1^3C_2 - \\
& 2R^2C_2^2 + 2C_2^2).
\end{aligned}$$

The initial conditions for the risk-sensitive filter are: $X_1(0) = 0{,}31$, $X_2(0) = 0{,}05$, $Y_1(0) = 0{,}85$, $C_1(0) = -1{,}9$ y $C_2(0) = 1{,}415$. The initial conditions for the $Q's$ equations are given in the Table I.

The system formed by equations (15), (16) and (17), is simulated using Simulink MatLab7. The design of the equations is compared against Polynomial Filtering equations (M. V. Basin and Darío Calderón Álvarez, 2009), applied to the system (15) and (16).

TABLA I

INITIAL CONDITIONS FOR THE EQUATIONS FOR R-S ESTIMATOR AND THE POLYNOMIAL FILTERING

R-S Estimator	Polinomial Filtering	
$Q_{11}(0) = -80$	$P_{11}(0) = 25$	$V = 1$
$Q_{12}(0) = -0{,}26$	$P_{12}(0) = 1 = P_{21}$	$R = 0{,}001$
$Q_{22}(0) = -60$	$P_{22}(0) = 20$	$T = 0{,}5s$

TABLA II

COMPARISON OF MEAN-SQUARE EXPONENTIAL CRITERION J (10) FOR R-S FILTER AND POLYNOMIAL FILTER.

ϵ	J_{R-S}	J_{Pol}
0,001	0,7051 $T = 0{,}3898$	0,7048 $T = 0{,}09$
0,01	0,8022	1,5112
0,1	0,8025	1,5049
1	0,8034	1,4694
10	0,8063	4,15
100	0,8159	49,7508
1000	0,8496	2907,427

IV-B. *Optimal Estimator for Polynomial States over Polynomial Observations*

The equations of the Optimal Estimator for polynomial states over polynomial observations are given ((M. V. Basin and Darío Calderón Álvarez, 2009)) by:

$$\begin{aligned}
\dot{m}_1 = {} & (1 - 2m_1)^{-1}[\sqrt{\tfrac{\epsilon}{2\gamma^2}} + P_{11}(dy_1 - m_1) + \\
& P_{12}(dy_2 - m_2)], \\
\dot{m}_2 = {} & (1 - 2m_2)^{-1}[\sqrt{\tfrac{\epsilon}{2\gamma^2}} + P_{21}(dy_1 - m_1) + \\
& P_{22}(dy_2 - m_2)], \\
\dot{P}_{11} = {} & (2V + 2\dot{m}_1)^2P_{11} + \tfrac{\epsilon}{2\gamma^2} - P_{11}^2 - P_{12}P21, \\
\dot{P}_{12} = {} & (2V + 2\dot{m}_1)(2V + 2\dot{m}_2)P_{12} - P_{11}P_{12} - P_{12}P22, \\
\dot{P}_{21} = {} & (2V + 2\dot{m}_2)(2V + 2\dot{m}_1)P_{21} - P_{11}P_{21} - P_{21}P22, \\
\dot{P}_{22} = {} & (2V + 2\dot{m}_2)^2P_{22} + \tfrac{\epsilon}{2\gamma^2} - P_{22}^2 - P_{21}P12.
\end{aligned}$$

Where the initial conditions are: $X_1(0) = 0{,}31$, $X_2(0) = 0{,}05$, $Y_1(0) = 0{,}85$, $m_1(0) = -2{,}85$ y $m_2(0) = -0{,}15$. The initial conditions for $P's$ are given in the Table I.

The systems of equations are simulated with the initial conditions of Table I.

Then we can see that the difference in the values of the criteria is large, since for the risk-sensitive estimator was smaller.

Table II presents some values of the risk-sensitive and Polynomial mean-square exponential criterion values, it can be observed that the J_{r-s} values are the smallest values.

The graphics 1 and 2 show the $Error_1$, which is defined as $E_1 = X_1(t) - C_1(s)$; the $Error_2$ is defined as $E_2 = X_2(t) - C_2(s)$, and the mean-square exponential criterion.

V. CONCLUSIONS

In this work we obtained the equations of the problems of optimal risk-sensitive estimator and the optimal filtering for polynomial systems over polynomial observations, when the system is a polynomial of degree one and the observations

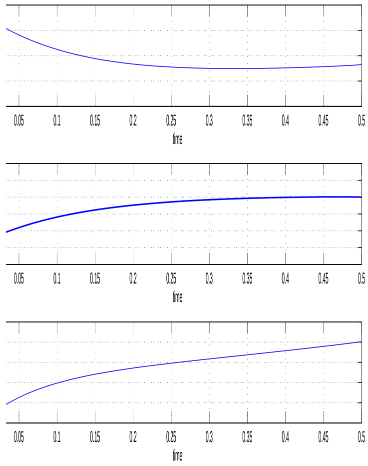

Figura 1. Error 1, Error 2 and R-S Criterion.

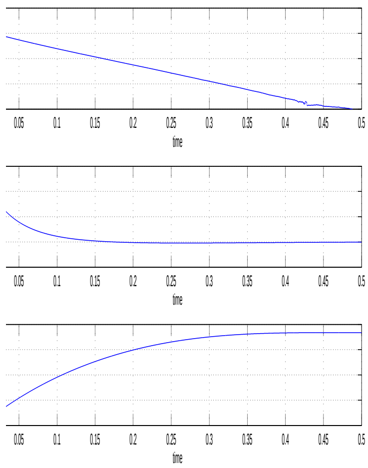

Figura 2. Error 1, Error 2 and Polynomial Criterion.

are second grade, with presence of white Gaussian noise, the exponential cost criterion mean- square was minimized, with the parameter $\tilde{\epsilon}$ multiplying the Gaussian white noise, and taking into account a value function as a viscous solution of partial differential equation (HJB).

A numerical application was determined for both cases (risk-sensitive filtering and filtering polynomial) for some values of the parameter $\tilde{\epsilon}$. The advantage of the equations of optimal risk-sensitive estimator is verified through the values of exponential cost criterion of mean square J with respect to the polynomial filter. The graphics show the difference between the state and the estimator obtained, also the graph of the exponential mean-square criterion.

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